

SOME RESULTS ON pth GOL'DBERG RELATIVE ORDER

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ABSTRACT

In this paper, we obtain some results on the pth Gol'dberg relative order of entire functions of several complex variables which improve some earlier results.

KEYWORDS: Entire Function, Relative Order, Gol'dberg Order, Gol'dberg Relative Order, Property(R)

INTRODUCTION

Let f and g be two non-constant entire functions and

 $M_f(r) = \max\{|f(z)|: |z| = r\}, M_g(r) = \max\{|g(z)|: |z| = r\}$ be the maximum modulus functions of f and g respectively. Then $M_f(r)$ is strictly increasing function and continuous of r and its inverse $M_f^{-1}: (|f(0)|, \infty) \to (0, \infty)$ exits and $\lim_{s\to\infty} M_f^{-1}(s) = \infty$.

Definition 1: The order ρ_f and lower order λ_f of an entire function f are defined as follows:

$$\rho_{f} = \lim_{r \to \infty} \sup \frac{\log^{[2]} M_{f}(r)}{\log r} \text{ and } \lambda_{f} = \lim_{r \to \infty} \inf \frac{\log^{[2]} M_{f}(r)}{\log r}$$

The function f is said to be of regular growth if $\rho_f = \lambda_f$.

In [1] Lahiri and Banerjee considered a more general definition of order as follows:

Definition 2: [1] if $p \ge 1$ is a positive integer, then the pth relative order of f with respect tog, denoted by $\rho_g^{[p]}(f)$ is defined as

$$\rho_{g}^{[p]}(f) = \inf \{ \mu > 0 : M_{f}(r) < M_{g}(exp^{[p-1]}r^{\mu}), \text{for all } r > r_{0}(\mu) > 0 \}$$

If p=1, g (z) =expz, then $\rho_g^{[p]}(f) = \rho_g(f)$ the classical order off.

Definition 3: [4] Let $f(z_1, z_2)$ and $g(z_1, z_2)$ are two non constant entire functions of two complex variables z_1 and z_2 holomorphic in the closed polydisc

$$\{(z_1, z_2): |z_i| \le r_i; i = 1,2\}$$
 And

Let
$$M_f(r_1, r_2) = \max\{|f(z)|: |z_i| = r_i; i = 1, 2\}, M_g(r_1, r_2) = \max\{|g(z_1, z_2)|: |z_i| \le r_i; i = 1, 2\}$$

The relative order of f with respect to g denoted $by\rho_g(f)$, is defined as

$$\rho_{g}(f) = \inf \left\{ \mu > 0 : M_{f}(r) < M_{g}(r_{1}{}^{\mu}, r_{2}{}^{\mu}), \text{ for all } r_{1} \ge R(\mu), r_{2} \ge R(\mu) \right\}.$$

We recall the following notation and definition of relative order of entire functions of n complex variables. We

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denote the point $(z_1, z_2, ..., z_n) \in \mathbb{C}^n$ and $(m_1, m_2, ..., m_n) \in I^n$ by z and m respectively I denote the set of all non negative integers. Where \mathbb{C}^n denote the n-dimensional complex space. $|z| = (|z_1|^2 + |z_2|^2 + \dots + |z_n|^2)^{1/2}$ also we can write $||m|| = m_1 + m_2 + \dots + m_n$. Let bounded complete $n - \text{circular domain } D \subseteq \mathbb{C}^n$ with centre at origin. Let

$$M_{f,D}(R) = \sup_{z \in D_R} |f(z)|$$

For R > 0 a point $z \in D_R$ iff $\frac{z}{R} \in D$ where f is an entire function of n complex variables Let g be a non constant entire function then $M_{g,D}(R)$ is strictly increasing continuous and its inverse

 $M_{g,D}^{-1}$: $(|g(0)|, \infty) \to (0, \infty)$ Exists such that $\lim_{R\to\infty} M_{g,D}^{-1}(R) = \infty$. The Gol'dberg order of entire function of n complex variables is defined as follows:

Definition 4: [7] The Gol'dberg order $\rho_{f,D}$ of f with respect to domain is defined as follows

$$\rho_{f,D} = lim_{R \rightarrow \infty} sup \frac{\log{^{[2]}M_{f,D}(R)}}{logR}$$

The lower Gol'dberg order $\lambda_{f,D}$ of f with respect to domain D is defined as

$$\lambda_{f,D} = \lim_{R \to \infty} \inf \frac{\log^{[2]} M_{f,D}(R)}{\log R}$$

If f is regular growth if $\rho_{f,D} = \lambda_{f,D}$.

The $\rho_{f,D}$, order is independent of the choice of the domain D in {cf. [7]} and therefore we denote the order of f as ρ_{f} .

Definition 5: p^{th} Gol'dberg order and lower Gol'dberg order are denoted by $\rho_{f,D}^{[p]}$ and $\lambda_{g,D}^{[p]}$ are respectively defined as follows:

$$\rho_{f,D}^{[p]} = \lim_{R \to \infty} \sup \frac{\log |p| M_{f,D}(R)}{\log R}$$

And
$$\lambda_{f,D}^{[p]} = \lim_{R \to \infty} \inf \frac{\log |p| M_{f,D}(R)}{\log R}$$

Where p = 2,3,4, ...

In recent paper Mondale and Roy [6] introduced the concept of relative order of entire functions of n-complex variables. They gave the following definition.

Definition 6: ([1],[4],[6]) Let f, and g be entire functions of n complex variables and D be a bounded complete ncircular domain with centre at the origin in Cⁿ. Then the pth relative order $\rho_{g,D}^{[p]}(f)$ of f with respect to g in the domain D is defined by

$$\begin{split} \rho_{g,D}^{[p]}(f) &= \inf\{\mu > 0: M_{f,D}(r) < M_{g,D}(exp^{[p-1]}R^{\mu}), \text{for } R \ge R_0(\mu) > 0\} \\ &= &\lim_{R \to \infty} sup \frac{\log^{[p]} M_{g,D}^{-1}(M_{f,D}(R))}{\log R} \end{split}$$

Where p = 1, 2, 3, ...

If we take $g(z) = e^{z} = e^{(z_1, z_2, ..., z_n)}$ and p = 1, then the relative Gol'dberg order $\rho_{g,D}(f)$ of f with respect to g in the domain D coincides with Gol'dberg $\rho_{f,D}$ of f with respect to domain D.

We define the pth Gol'dberg relative order $\lambda_{g,D}^{[p]}(f)$ of f with respect to g in the domain as

$$\lambda_{g,D}^{[p]}(f) = \text{lim}_{R \rightarrow \infty} \text{inf} \frac{\log{^{[p]}M_{g,D}^{-1}(M_{f,D}(R))}}{\log R}$$

Where p = 1, 2, 3, ...

In this paper we obtain some relationship between relative order, relative lower order, Gol'dberg order and Gol'dberg lower order, pth Gol'dberg relative (order and lower order) of entire functions of several complex variables which improves some earlier results.

THEOREMS

Theorem 1: Let f and g be entire functions of n complex variables such that $0 < \lambda_f^{[p]} \le \rho_f^{[p]}$ and $0 < \lambda_g^{[p]} \le \rho_g^{[p]}$. Then

$$\lambda_{f}^{[p]} / \rho_{g}^{[p]} \leq \lambda_{g}^{[p]}(f) \leq \min \left\{ \lambda_{f}^{[p]} / \lambda_{g}^{[p]}, \rho_{f}^{[p]} / \rho_{g}^{[p]} \right\} \leq \max \left\{ \lambda_{f}^{[p]} / \lambda_{g}^{[p]}, \rho_{f}^{[p]} / \rho_{g}^{[p]} \right\} \leq \rho_{g}^{[p]}(f) \leq \frac{\rho_{f}^{[p]} / \lambda_{g}^{[p]}}{\lambda_{g}^{[p]}}$$

Proof: From the definition of p^{th} -Gol'dberg order and lower order we get arbitrary $\epsilon > 0$ and for all large values of R then

$$M_{f}(R) < exp\left(exp^{[p-1]}R^{\left(\rho_{f}^{[p]}+\varepsilon\right)}\right)$$
(1)

$$M_{g}(R) < exp\left(exp^{[p-1]}R^{\left(\rho_{g}^{[p]}+\varepsilon\right)}\right)$$
(2)

$$M_{f}(R) > exp\left(exp^{[p-1]}R^{\left(\lambda_{f}^{[p]}-\varepsilon\right)}\right)$$
(3)

$$M_{g}(R) > exp\left(exp^{[p-1]}R^{\left(\lambda_{g}^{[p]}-\varepsilon\right)}\right)$$
(4)

Also for a sequence $\{R_n\}$ tending to infinity we get

$$M_{f}(R_{n}) > exp\left(exp^{[p-1]}R_{n}^{\left(\rho_{f}^{[p]}-\varepsilon\right)}\right)$$
(5)

$$M_{g}(R_{n}) > exp\left(exp^{[p-1]}R_{n}^{\left(\rho_{g}^{[p]}-\varepsilon\right)}\right)$$
(6)

$$M_{f}(R_{n}) < exp\left(exp^{[p-1]}R_{n}^{\left(\lambda_{f}^{[p]}+\varepsilon\right)}\right)$$

$$\tag{7}$$

And
$$M_{f}(R_{n}) < exp\left(exp^{[p-1]}R_{n}^{(\lambda_{g}^{[p]}+\varepsilon)}\right)$$
(8)

Now from the definition of p^{th} relative order we get, for arbitrary $\epsilon_1 > 0$ and for all large values of R then

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$$\rho_g^{[p]}(f) + \epsilon_1 > \frac{\log^{[p]} M_g^{-1}(M_f(R))}{\log R}$$

Now from (5) we get for a sequence $\{R_n\}$ tending to infinity that

$$\begin{split} \rho_{g}^{[p]}(f) + \epsilon_{1} &> \frac{\log^{[p]} M_{g}^{-1} \left(\exp\left(\exp^{[p-1]} R_{n}^{\left(\rho_{f}^{[p]} - \epsilon\right)} \right) \right)}{\log R_{n}} \\ &= \frac{\log^{[p]} M_{g}^{-1} \left(\exp\left(\exp^{[p-1]} R_{n}^{\left(\frac{\rho_{f}^{[p]} - \epsilon}{\rho_{g}^{[p]} + \epsilon} \right)} \right)}{\log R_{n}} \\ &= \frac{\log^{[p]} M_{g}^{-1} M_{g} \left(\exp^{[p-1]} R_{n}^{\left(\frac{\rho_{f}^{[p]} - \epsilon}{\rho_{g}^{[p]} + \epsilon} \right)} \right)}{\log R_{n}} \\ &> \frac{\log^{[p]} M_{g}^{-1} M_{g} \left(\exp^{[p-1]} R_{n}^{\left(\frac{\rho_{f}^{[p]} - \epsilon}{\rho_{g}^{[p]} + \epsilon} \right)} \right)}{\log R_{n}} \end{split}$$

As $\epsilon_1>0$ and $\epsilon>0$ are arbitrary we get that

$$\rho_{g}^{[p]}(f) \ge \frac{\rho_{f}^{[p]}}{\rho_{g}^{[p]}}$$

$$\tag{9}$$

Also from (1) we get for arbitrary $\epsilon > 0$ and for all large values of R that

$$\begin{split} \frac{\log^{[p]} M_g^{-1}(M_f(R))}{\log R} &< \frac{\log^{[p]} M_g^{-1} \left(\exp\left(\exp\left(\exp^{[p-1]} R^{\left(\rho_f^{[p]} + \epsilon\right)} \right) \right)}{\log R} \\ & = \frac{\log^{[p]} M_g^{-1} \left(\exp\left(\exp\left(\exp^{[p-1]} R^{\left(\frac{\rho_f^{[p]} + \epsilon}{\rho_g^{[p]} - \epsilon} \right)} \right) \frac{1}{\rho_g^{[p]} - \epsilon} \right)}{\log R} \end{split}$$

Now from (6) we get for a sequence $\{R_n\}$ tending to infinity that

$$\begin{split} \frac{\log^{[p]} M_g^{-1}(M_f(R_n))}{\log R_n} & < \frac{\log^{[p]} M_g^{-1} M_g \left(\exp^{[p-1]} R_n^{\left(\frac{\rho_f^{[p]} + \epsilon}{\rho_g^{[p]} - \epsilon} \right)}{\log R_n} \right)}{\log R_n} \end{split}$$

As $\varepsilon > 0$ is arbitrary we have

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$$\lambda_{g}^{[p]}(f) \le \frac{\rho_{f}^{[p]}}{\rho_{g}^{[p]}} \tag{10}$$

Now from the definition of p^{th} relative lower order we get for arbitrary $\epsilon_2 > 0$ and for all large values of R that

$$\lambda_g^{[p]}(f) - \epsilon_2 < \frac{\log^{[p]} \mathsf{M}_g^{-1}(\mathsf{M}_f(\mathsf{R}))}{\log \mathsf{R}}$$

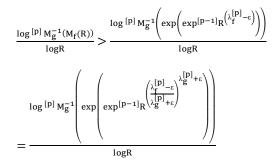
Now from (7) we get for a sequence $\{R_n\}$ tending to infinity

$$\begin{split} \lambda_{g}^{[p]}(f) &- \epsilon_{2} < \frac{\log^{[p]} M_{g}^{-1} \left(\exp\left(\exp^{[p-1]} R_{n}^{\left(\lambda_{f}^{[p]} + \epsilon\right)} \right) \right)}{\log R_{n}} \\ &= \frac{\log^{[p]} M_{g}^{-1} \left(\exp\left(\exp^{[p-1]} R_{n}^{\left(\frac{\lambda_{f}^{[p]} + \epsilon}{\lambda_{g}^{[p]} - \epsilon} \right)} \right) \right)}{\log R_{n}} \\ &< \frac{\log^{[p]} M_{g}^{-1} M_{g} \left(\exp^{[p-1]} R_{n}^{\left(\frac{\lambda_{f}^{[p]} + \epsilon}{\lambda_{g}^{[p]} - \epsilon} \right)} \right)}{\log R_{n}} \\ &= \frac{\lambda_{f}^{[p]} + \epsilon}{\lambda_{g}^{[p]} - \epsilon} \end{split}$$

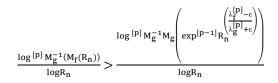
As $\varepsilon_2 > 0$ and $\varepsilon > 0$ are arbitrary we obtain that

$$\lambda_{g}^{[p]}(f) \le \frac{\lambda_{f}^{[p]}}{\lambda_{g}^{[p]}}$$
(11)

Now from (3) we get arbitrary $\varepsilon > 0$ and for all large values of R that

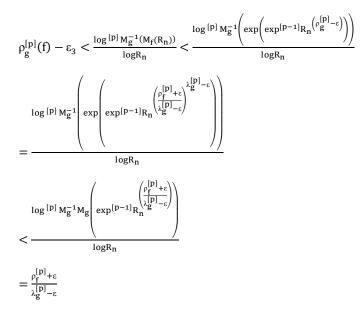


Now from (8) we obtain for a sequence $\{R_n\}$ tending to infinity that



$$\lim_{R_n \to \infty} \sup \frac{\log^{[p]} M_g^{-1}(M_f(R_n))}{\log R_n} \ge \frac{\lambda_f^{[p]} - \varepsilon}{\lambda_g^{[p]} + \varepsilon}$$

As $\varepsilon > 0$ is arbitrary $\varepsilon_3 > 0$ and for a sequence $\{R_n\}$ tending to infinity that



 $\epsilon_3>0$ And $\epsilon>0$ are arbitrary, we have

$$\rho_{g}^{[p]}(f) \leq \frac{\rho_{f}^{[p]}}{\lambda_{g}^{[p]}}$$

$$\tag{13}$$

As we get for arbitrary $\ \epsilon_4>0$ and $\epsilon>0\,$ for a $\{R_n\}$ tending to infinity that

$$\begin{split} \lambda_{g}^{[p]}(f) + \varepsilon_{4} &> \frac{\log^{[p]} M_{g}^{-1}(M_{f}(R_{n}))}{\log R_{n}} > \frac{\log^{[p]} M_{g}^{-1} \left(\exp\left(\exp^{[p-1]} R_{n}^{\left(\lambda_{g}^{[p]}-\varepsilon\right)}\right) \right)}{\log R_{n}} \\ &= \frac{\log^{[p]} M_{g}^{-1} \left(\exp\left(\exp^{[p-1]} R_{n}^{\left(\frac{\lambda_{f}^{[p]}-\varepsilon}{\rho_{g}^{[p]}+\varepsilon}\right)}\right)^{\rho_{g}^{[p]}+\varepsilon} \right)}{\log R_{n}} \\ &< \frac{\log^{[p]} M_{g}^{-1} M_{g}^{\left(\exp^{[p-1]} R_{n}^{\left(\frac{\lambda_{f}^{[p]}-\varepsilon}{\rho_{g}^{[p]}+\varepsilon}\right)}\right)}{\log R_{n}} \\ &< \frac{\log^{[p]} M_{g}^{-1} M_{g}^{\left(\exp^{[p-1]} R_{n}^{\left(\frac{\lambda_{f}^{[p]}-\varepsilon}{\rho_{g}^{[p]}+\varepsilon}\right)}\right)}{\log R_{n}} \end{split}$$

As $\epsilon_4 > 0$ and $\epsilon > 0$ are arbitrary, we obtain that

$$\lambda_{\mathbf{g}}^{[\mathbf{p}]}(\mathbf{f}) \ge \frac{\lambda_{\mathbf{f}}^{[\mathbf{p}]}}{\rho_{\mathbf{g}}^{[\mathbf{p}]}} \tag{14}$$

The theorem follows from (9), (10), (11), (12), (13) and (14).

Corollary 1: If both f and g are regular growth with non zero order then

$$\rho_{g}^{[p]}(f) = \rho_{f}^{[p]}(g) \text{ iff} \rho_{g}^{[p]} = \rho_{f}^{[p]}.$$

Corollary 2: If both f and g are regular growth with non zero order then

$$\lambda_{g}^{[p]}(f) = \rho_{g}^{[p]}(f)$$

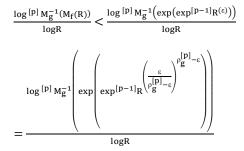
Corollary 3: If g is regular growth with non zero order then

$$\rho_{g}^{[p]}(f) \ge \frac{\rho_{f}^{[p]}}{\rho_{g}^{[p]}}$$

Theorem 2: Let f and g be entire functions of n complex variables such that $\rho_f^{[p]} = 0$ and $0 < \rho_g^{[p]} < \infty$. Then $\lambda_g^{[p]}(f) = 0$

Proof: From the definition of Gol'dberg order we have for arbitrary $\varepsilon > o$ and for all large values of r that

$$M_{f}(R) < exp(exp^{[p-1]}R^{(\varepsilon)})$$



Now from (6) we get for a sequence $\{R_n\}$ tending to infinity that

$$\frac{\log^{[p]} M_g^{-1}(M_f(R_n))}{\log R_n} < \frac{\log^{[p]} M_g^{-1} M_g \left(\exp^{[p-1]} R_n^{\left(\frac{\epsilon}{\rho[p]}\right)} \right)}{\log R_n}$$
$$\lim_{R_n \to \infty} \inf \frac{\log^{[p]} M_g^{-1}(M_f(R_n))}{\log R_n} \le \frac{\epsilon}{\rho_r^{[p]} - \epsilon}$$

As $\epsilon>0$ is arbitrary it follows that $\lambda_g^{[p]}(f)=0.$

Theorem 3: Let f and g be entire functions of n complex variables such that $0 < \rho_f^{[p]} < \infty$ and $\rho_g^{[p]} = 0$. Then $\rho_g^{[p]}(f) = \infty$

Proof: From the definition of relative order we get for arbitrary $\varepsilon_1 > 0$ and for all large values of R that

$$\rho_{g}^{[p]}(f) + \epsilon_{1} > \frac{\log^{[p]} M_{g}^{-1}(M_{f}(R))}{\log R}$$

Now from (5) we get for a sequence $\{R_n\}$ tending to infinity

$$\begin{split} \rho_{g}^{[p]}(f) + \varepsilon_{1} &> \frac{\log^{[p]} M_{g}^{-1} \left(\exp\left(\exp^{[p-1]} R_{n}^{\left(\rho_{f}^{[p]} - \varepsilon\right)} \right) \right)}{\log R_{n}} \\ &= \frac{\log^{[p]} M_{g}^{-1} \left(\exp\left(\exp^{[p-1]} R_{n}^{\left(\frac{\rho_{f}^{[p]} - \varepsilon}{\varepsilon}\right)^{\varepsilon}} \right) \right)}{\log R_{n}} \\ &< \frac{\log^{[p]} M_{g}^{-1} M_{g} \left(\exp^{[p-1]} R_{n}^{\left(\frac{\rho_{f}^{[p]} - \varepsilon}{\varepsilon}\right)} \right)}{\log R_{n}} \\ &= \frac{\rho_{f}^{[p]} - \varepsilon}{\varepsilon} \end{split}$$

As $\epsilon_1>0$ and $\epsilon>0$ are arbitrary it follows that $\rho_g^{[p]}(f)=\infty.$

ACKNOWLEDGEMENTS

The author is thankful to the referee for her valuable suggestion towards the improvement of the paper.

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